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1987 J. Phys. A: Math. Gen. 20 3667

(http://iopscience.iop.org/0305-4470/20/12/020)

J. Phys. A: Math. Gen. **20** (1987) 3667–3678. Printed in the UK

Angular momentum balance and transverse shifts on reflection of light

M A Player

Department of Physics, University of Aberdeen, Aberdeen AB9 2UE, UK

Received 7 August 1986, in final form 28 January 1987

Abstract. Many years ago it was pointed out by Holbourn that there is an apparent breakdown of angular momentum conservation on reflection of circularly polarised light. The present paper points out that the discrepancy may be resolved by considering transverse positional shifts on reflection as described by Schilling, Imbert and subsequent authors, and examines the values of the shifts necessary to satisfy angular momentum conservation in various situations. It is found that shifts satisfactory in this respect may be obtained by use of a fairly general phase-shift argument—similar in formalism to that used by Julia and Neveu—at least for the case of total reflection. However, the situation is less satisfactory for partial reflection, nor is it clear that the shift values, even for the case of total reflection, agree with those measured by Imbert and others, or calculated numerically as by Ashby and Miller for instance. It is suggested that further experimental work may be useful in clarifying the situation.

1. Introduction

The change in angular momentum of light on reflection has been discussed by relatively few workers, despite the appearance in this connection of certain difficulties which were pointed out long ago by Holbourn (1936).

These difficulties arise from the observation that there can be no normal component of torque produced at a non-absorbing surface by reflection of light. This conclusion can be reached by a *reductio ad absurdum*: if there were such a torque present, then rotation of the surface about its normal (i.e. in its own plane) would do work on the light, changing its frequency, and this would lead to the phase of the outgoing beam being dependent on the angle through which the surface is turned about its normal; since there is actually full rotational symmetry of the reflecting surface about its normal, this result is absurd and the torque must therefore in fact be zero. Hence, the flux of normal component of angular momentum of the light, across a surface surrounding the region of reflection, is also required to be zero. For an absorbing surface the argument does not apply, since in this case the rotation invoked in the *reductio ad absurdum* may do work on the surface, via the dissipative forces, rather than on the light.

In view of this argument, Holbourn examined the case of external reflection from a simple dielectric surface and found a non-zero net flux of the normal component of intrinsic angular momentum. To show this, consider a circularly polarised incident beam with energy flux F_i (and hence intrinsic angular momentum flux F_i/ω, independent of the refractive index of the medium). If the angle of incidence is i, then this angular momentum flux has a component normal to the reflecting surface which is just $(F_i/\omega)\cos i$, measured per unit area of the beam. Throughout this paper, we shall use quantities normalised to unit area of illuminated interface; since the ratio of illuminated area to beam area is just $1/\cos i$, the flux of normal component of angular momentum, per unit area of illuminated interface, is $(F_i/\omega)\cos^2 i$, for the incident beam. Treating the contributions from the reflected and transmitted beams similarly gives for the net flux S' of normal component of angular momentum across unit area of the illuminated interface:

$$S' = \frac{F_i}{\omega}[\cos^2 i + (|\rho_+|^2 - |\rho_-|^2)\cos^2 i - (|\tau_+|^2 - |\tau_-|^2)\cos^2 i'] \tag{1.1}$$

where i' is the angle of refraction, ρ_+ and ρ_- are the amplitudes in the reflected beam for light polarised, respectively, like and orthogonal to the circularly polarised incident beam and τ_+ and τ_- are similar quantities for the transmitted beam (all measured relative to the incident beam amplitude, and also calculated so that in each case the squared modulus is equal to the relative energy flux of the relevant component of the beam). Take now for illustration the case of incidence from a medium of refractive index 1, onto a medium of index n. As discussed further in § 3, the amplitudes for reflection and transmission of circularly polarised components are related to the usual reflection and transmission ratios $r_\parallel$, $r_\perp$, $t_\parallel$, $t_\perp$ for linearly polarised components (with electric vector parallel and perpendicular to the plane of incidence) by

$$\begin{aligned} \rho_+ &= \tfrac{1}{2}(r_\parallel + r_\perp) \qquad & \rho_- &= \tfrac{1}{2}(r_\parallel - r_\perp) \\ \tau_+ &= \tfrac{1}{2}\sqrt{n}(t_\parallel + t_\perp) \qquad & \tau_- &= \tfrac{1}{2}\sqrt{n}(t_\parallel - t_\perp) \end{aligned} \tag{1.2}$$

in which the factor $\sqrt{n}$ is included so that $|\tau_+|^2$ and $|\tau_-|^2$ give the relative energy fluxes in the transmitted beam components. If we further specialise for simplicity to the case of Brewster angle incidence, then Fresnel's equations give

$$\begin{aligned} r_\parallel &= 0 \qquad & r_\perp &= (1-n^2)/(1+n^2) \\ t_\parallel &= 1/n \qquad & t_\perp &= 2/(1+n^2) \end{aligned} \tag{1.3}$$

and substituting these values in equation (1.2) we find from equation (1.1) that S' has the non-zero value

$$S' = \frac{F_i}{\omega}\frac{1-n^2}{(1+n^2)^2}. \tag{1.4}$$

It is clear that, even had S' been zero in this particular case, one could envisage coated surfaces with more or less arbitrary values of reflectance for which S' certainly would not vanish. Similar remarks of course apply for an arbitrary input beam, although here the detailed results are rather less simple in general. We may, however, use an argument similar to that for equation (1.1) to conclude that in general the flux of normal component of intrinsic angular momentum, per unit area of illuminated interface, is

$$S' = \frac{F_i}{\omega}[(I_+ - I_-)\cos^2 i + (R_+ - R_-)\cos^2 i - (T_+ - T_-)\cos^2 i'] \tag{1.5}$$

where I_+, I_-, R_+, R_-, T_+, T_- denote the relative intensitities (i.e. energy fluxes) of the right- and left-handed circularly polarised components of incident, reflected and transmitted beams, with the normalisation

$$I_+ + I_- = 1. \tag{1.6}$$

The present paper will examine an additional source of angular momentum flux, associated with positional shifts of the reflected and transmitted beams, which, combined with the flux of intrinsic angular momentum expressed by equation (1.5), can give zero net flux as required for conservation of angular momentum. It will be demonstrated that the values of positional shifts apparently required for angular momentum balance are consistent with some, but not all, of the values reported in the literature. In addition, a relatively simple argument (partly following Julia and Neveu (1973)) will be developed and shown to give positional shifts consistent with correct angular momentum balance in a variety of situations, including total reflection and at least some cases of partial reflection.

2. Transverse positional shifts and angular momentum

Transverse (or lateral) shifts in position of light beams undergoing reflection, in cases where the beams have elliptical polarisation, are now well known and indeed have caused considerable interest for some time (see, for instance, Costa de Beauregard and Imbert 1973, Julia and Neveu 1973, Ashby and Miller 1976). Such shifts, in which there is a positional displacement between the incident and reflected beams which is directed normal to the plane of incidence, arise in a fashion similar to the longitudinal positional shifts described earlier by Goos and Hänchen (see, for example, Lotsch 1970a, b). The transverse positional shifts, combined with optical radiation pressure, provide a source of angular momentum flux which was neglected by Holbourn.

In order to derive a value for this additional flux of angular momentum, consider the general case of partial reflection in which the reflected and transmitted beams are both displaced from the incident beam; the components of this displacement normal to the plane of incidence will be written S_r and S_t, respectively, for the two beams, and will be referred to as the transverse positional shifts. If n and n' are the refractive indices for incidence medium and transmission medium, respectively, then the reflected beam will carry an effective flux of momentum $(n/c)F_i(R_+ + R_-)$, with a component $(n/c)F_i(R_+ + R_-)\sin i$ parallel to the plane of the reflecting surface; the corresponding component for the transmitted beam is $(n'/c)F_i(T_+ + T_-)\sin i'$. The refractive indices in these expressions account for the observed behaviour of radiation pressure in a refractive medium (Jones and Leslie 1978). Taking moments about the centre of the incident beam, these radiation pressures contribute a torque about the normal to the surface, per unit area of illuminated interface, given by

$$L' = \frac{F_i}{c}[n(R_+ + R_-)S_r \cos i \sin i + n'(T_+ + T_-)S_t \cos i' \sin i'] \qquad (2.1)$$

in which the factors $\cos i$ and $\cos i'$ arise from the normalisation to unit area of the interface, as in the argument leading to equation (1.1). L' thus gives the additional component of flux of angular momentum. Conservation of the component of linear momentum parallel to the reflecting interface ensures that the value of L' is actually independent of the point about which moments are taken.

The above argument could be criticised on the basis of the value taken for radiation pressure in a medium. As an alternative, one can discuss the torque on a pair of cones having a common axis and fixed base to base, with one cone of each medium. Then, the plane interface constitutes the reflecting surface, and as a simplification the conical surfaces may be antireflection treated and chosen normal to the beams traversing them.

Such an object should suffer zero net torque by the argument given in § 1 and the value of L' appropriate to this case is again given as above, for the beam momenta may all be calculated *in vacuo*, and the factors n, n' now arise through a change in refraction angle at each conical surface due to the relevant displacement.

Hence, on either argument, the shift-dependent part of the flux of normal component of angular momentum is, using Snell's law in equation (2.1),

$$L' = n\frac{F_{\rm i}}{c}\sin i[(R_+ + R_-)S_r\cos i + (T_+ + T_-)S_t\cos i'] \tag{2.2}$$

in which n refers to the incidence medium.

Now the question arises of what values to employ for the shifts. The case which has been most extensively studied is that of total internal reflection of circularly polarised light at near critical incidence, for which the state of polarisation of the incident light is unchanged on reflection. In this situation one has just $I_+ = 1$, $I_- = 0$, $R_+ = 1$, $R_- = 0$, $T_+ = 0$, $T_- = 0$ and, substituting these values in equations (1.5) and (2.2), respectively, gives

$$S' = 2\frac{F_{\rm i}}{\omega}\cos^2 i \tag{2.3}$$

$$L' = n\frac{F_{\rm i}}{c}\cos i\sin i\, S_r. \tag{2.4}$$

Substitution of an actual value of transverse positional shift S_r in (2.4) now allows the total angular momentum flux $S' + L'$ to be obtained. If angular momentum balance is required, this analysis therefore favours the expression for transverse positional shift

$$S_r = 2\frac{c}{n\omega}\cot i \tag{2.5}$$

obtained by Schilling (1965) using a phase shift argument, and not the form

$$S_r = 2\frac{c}{n\omega}\frac{1}{\cos i\sin i} \tag{2.6}$$

suggested by Ricard (1970) and Imbert (1972) on energy flux considerations.

The arguments used by the cited workers in obtaining the conflicting values for S_r exemplify well the two main methods used to evaluate both the longitudinal and transverse positional shifts. Phase shift arguments essentially use an approach from Fourier optics: the beams are considered as a superposition of plane wave components with wavevectors of differing directions. If these components undergo a process which introduces a relative phase shift depending on wavevector direction, the structure of the beam is altered; in particular, a phase shift proportional to wavevector angle is equivalent to a change in beam position. Energy flux methods consider the detailed energy balance in the optical beams, using the Poynting vector to calculate energy fluxes: for instance, in the case of total internal reflection, calculation of longitudinal and transverse components of the Poynting vector in the evanescent wave is used to derive values of the longitudinal and transverse positional shifts. Imbert (1972) points out the discrepancy between the results of the two approaches, but does not identify a specific error in Schilling's analysis.

The work of Imbert (1972) and Costa de Beauregard and Imbert (1973) included experimental verification of the energy flux result (equation (2.5)); however, these experiments have subsequently been criticised, particularly by Julia and Neveu (1973) and Ashby and Miller (1976). These workers all use phase shift arguments. Julia and Neveu obtain results in agreement with the experiments, but their argument involves the conclusion that Costa de Beauregard and Imbert actually measured something other than a simple transverse positional shift. Ashby and Miller point out that the reflection phase shifts are rapidly varying near the critical angle, and seek to take this into account by performing numerical calculations on actual wavepackets. Thus no analytical expression for the transverse shifts is obtained and, although the difference between shifts for right- and left-hand polarisations is roughly equal to the simple phase shift prediction, it is found that both polarisations are actually shifted in the same direction. This remarkable result clearly does not lead to a simple picture of angular momentum balance.

In view of all this, it appears to be of interest to see just how well a simple analytical phase shift argument agrees with the requirement of angular momentum balance. It will be particularly desirable to obtain results which are relatively insensitive to the detailed properties of the surface, so that the case of multilayer coatings can be included. Such a treatment is, of course, unlikely to handle detailed modelling of the structure of particular reflected wavepackets, but one hopes it may lead to reliable average values of the shifts.

3. Total internal reflection

We shall first reconsider the case of total internal reflection. In applying a phase shift argument, it is necessary to decide the form of beam to be constructed. Consider a beam in which all component plane waves have wavevectors with the same value of wavevector component normal to the reflection interface; this will enable us to study the transverse shifts for images of two-dimensional objects whose planes contain the normal to the interface, and which have no variation in structure in the direction of the normal. Such a restriction does not seem to be unduly severe and has the advantages that, since all beam components have the same angle of incidence, variations of reflection phase shifts with angle of incidence do not enter the problem. All that does enter is the relative rotation of vectors lying in the incidence planes, and normal to the wavevectors. Such rotations correspond to phase shifts of circularly polarised components of the beams and lie at the basis of the phase shift description of the transverse displacement.

It is in fact quite convenient to represent the state of polarisation, amplitude and phase of each beam by circular components, related to the standard Jones representation (see, for example, Shurcliff 1962) as follows. In the standard representation, the state of a (completely polarised) beam is represented by the vector

$$\begin{bmatrix} \alpha_{\parallel} \\ \alpha_{\perp} \end{bmatrix}$$

in which $a_{\parallel}$ and $a_{\perp}$ are the complex amplitudes of components of the electric vector parallel and perpendicular to a reference axis (which must be normal to the beam and, for the case discussed here, will be taken in the plane of incidence). In this representation circularly polarised beams have the Jones vectors:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \mathrm{j} \end{bmatrix} \qquad \text{and} \qquad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -\mathrm{j} \end{bmatrix}$$

($\mathrm{j}=\sqrt{-1}$ has been used to avoid confusion with the angle of incidence i). We now introduce the complex amplitudes of circular components a_+ and a_- by writing

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ \mathrm{j}\end{bmatrix}a_+ + \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -\mathrm{j}\end{bmatrix}a_- = \begin{bmatrix}a_\parallel\\ a_\perp\end{bmatrix}$$

from which it follows that the circular components are related to the linear components of the standard representation by

$$\begin{bmatrix}a_+\\ a_-\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -\mathrm{j}\\ 1 & \mathrm{j}\end{bmatrix}\begin{bmatrix}a_\parallel\\ a_\perp\end{bmatrix}. \tag{3.1}$$

Then the plane wave component of the reflected beam which has its wavevector along the nominal beam direction (the central component) will be related to the incident beam by

$$\begin{bmatrix}r_+\\ r_-\end{bmatrix} = \begin{bmatrix}\rho_+ & \rho_-\\ \rho_- & \rho_+\end{bmatrix}\begin{bmatrix}i_+\\ i_-\end{bmatrix} \tag{3.2}$$

where i_+, i_-, r_+, r_- are the circular components of the incident and reflected beams, respectively, and where comparison of the linear and circular representations (using a development similar to that leading to equation (3.1)) shows that

$$\rho_+ = \tfrac{1}{2}(r_\parallel + r_\perp) \qquad \rho_- = \tfrac{1}{2}(r_\parallel - r_\perp) \tag{3.3}$$

(which has already been used in equation (1.2)). It will be noted that for total internal reflection the transformation matrix in equation (3.2) is unitary. Now, for a plane wave component of the beam whose wavevector makes an angle θ with the central component, one may write

$$\begin{aligned}\begin{bmatrix}r_+\\ r_-\end{bmatrix}_{(\theta)} &= \begin{bmatrix}\exp(\mathrm{j}\varphi_r) & 0\\ 0 & \exp(-\mathrm{j}\varphi_r)\end{bmatrix}\begin{bmatrix}\rho_+ & \rho_-\\ \rho_- & \rho_+\end{bmatrix}\begin{bmatrix}\exp(\mathrm{j}\varphi_i) & 0\\ 0 & \exp(-\mathrm{j}\varphi_i)\end{bmatrix}\begin{bmatrix}i_+\\ i_-\end{bmatrix}\\ &= \begin{bmatrix}\exp[\mathrm{j}(\varphi_i+\varphi_r)]\rho_+ & \exp[-\mathrm{j}(\varphi_i-\varphi_r)]\rho_-\\ \exp[\mathrm{j}(\varphi_i-\varphi_r)]\rho_- & \exp[-\mathrm{j}(\varphi_i+\varphi_r)]\rho_+\end{bmatrix}\begin{bmatrix}i_+\\ i_-\end{bmatrix}\end{aligned} \tag{3.4}$$

in which the phase shifts φ_i, φ_r are functions of θ and express the effect of the rotations mentioned above, but ρ_+ and ρ_- remain constant. Again, the transformation matrix is unitary for total internal reflection. Now, the geometry enables φ_i and φ_r to be expressed in terms of θ. Consider two plane wave components of the incident beam, whose wavevectors lie in planes of incidence inclined at an angle ψ, and consider an electric vector for each wave lying in the respective plane of incidence (and of course normal to the respective wavevector). Use of elementary vector algebra shows that, for small angle ψ, the angle between the wavevectors is $\theta = \psi \sin i$, while the angle between the electric vectors is $\varphi = \psi \cos i$. This angle φ is interpreted as the phase shift between circularly polarised components of the two plane waves concerned and, on this interpretation, one obtains

$$\varphi_i = \varphi_r = \theta \cot i \tag{3.5}$$

so that, from (3.4) and (3.5)

$$\begin{bmatrix}r_+\\ r_-\end{bmatrix}_{(\theta)} = \begin{bmatrix}\exp(\mathrm{j}\varphi_r)\rho_+ & \rho_-\\ \rho_- & \exp(-\mathrm{j}\varphi_r)\rho_+\end{bmatrix}\begin{bmatrix}i_+\\ i_-\end{bmatrix} \tag{3.6}$$

with

$$\varphi_+ = 2\theta \cot i. \tag{3.7}$$

By using the explicit forms $r_\parallel = \exp(j\delta_\parallel)$ and $r_\perp = \exp(j\delta_\perp)$, and using the transformation equation (3.1), it may be verified that this corresponds to the transformation matrix A written by Julia and Neveu in the linear representation. In the case they consider, however, the geometry determines that φ_+ (their angle φ) has a different dependence on θ, and that ρ_+ and ρ_- also depend on θ.

Returning to our case, for a circularly polarised input beam of unit intensity ($i_+ = 1$, $i_- = 0$), the output beam will have the form

$$\begin{bmatrix} r_+ \\ r_- \end{bmatrix}_{(\theta)} = \exp(j\varphi_+)\begin{bmatrix} \rho_+ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \rho_- \end{bmatrix} \tag{3.8}$$

and will appear to consist of two circularly polarised components, of which only that polarised in the same sense as the input beam suffers a transverse displacement. Introducing the usual relation between transverse positional shift and rate of change of phase:

$$S_r = \frac{c}{n\omega}\dot{\varphi} \tag{3.9}$$

(where the dot represents differentiation with respect to θ) allows the shift of each polarisation component to be obtained. An 'average shift' may now be defined as the average of these two individual shifts, weighted by the relative intensities of the polarisation components; consideration of the derivation of equation (1.5) shows that this average value is what is required for the values of S_r and S_t. Using (3.9), the average shift for the beam represented in equation (3.8) is

$$S_r = 2\frac{c}{n\omega}|\rho_+|^2 \cot i. \tag{3.10}$$

For the case of a general input beam, the output beam

$$\begin{bmatrix} r_+ \\ r_- \end{bmatrix}_{(\theta)} = \begin{bmatrix} \exp(j\varphi_+)\rho_+ i_+ + \rho_- i_- \\ \rho_- i_+ + \exp(-j\varphi_+)\rho_+ i_- \end{bmatrix} \tag{3.11}$$

shows a less simple form. However, if we assume the beam can be decomposed into orthogonal components each showing a simple phase shift such that $\dot{\varphi}$ is constant across the plane wave components of the beam, then the average value of shift may, by an argument similar to that for equation (3.10), be written

$$\begin{aligned} S_r &= -j\frac{c}{n\omega}[r_1^* r_2^*]\begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \end{bmatrix} \\ &= -j\frac{c}{n\omega}[i_1^* i_2^*]T^\dagger \dot{T}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \end{aligned} \tag{3.12}$$

where r_1, r_2, i_1, i_2 represent the beams in terms of the appropriate shift eigenstates, and where T is the transformation matrix (appearing in equation (3.6)) written in the new representation. Thus the matrix $-j(c/n\omega)T^\dagger \dot{T}$ acts as an operator representing the transverse positional shift. In the case that T is unitary, the shift operator is Hermitian, the shift eigenstates are orthogonal and the shifts take real values, as desired. These points were recognised by Julia and Neveu. Hence to obtain the average shift, we may proceed in the original representation and find (at $\theta = 0$)

$$S_r = -j\frac{c}{n\omega}[i_+^* i_-^*]T^\dagger \dot{T}\begin{bmatrix} i_+ \\ i_- \end{bmatrix}$$

$$= \dot{\varphi}_+ \frac{c}{n\omega}[i_+^* i_-^*]\begin{bmatrix} |\rho_+|^2 & -\rho_-^*\rho_+ \\ \rho_-^*\rho_+ & -|\rho_+|^2 \end{bmatrix}\begin{bmatrix} i_+ \\ i_- \end{bmatrix}$$

$$= 2\frac{c}{n\omega}\cot i\,[|\rho_+|^2(|i_+|^2 - |i_-|^2) - \rho_+\rho_-^*(i_+^* i_- - i_+ i_-^*)]. \tag{3.13}$$

This may be put into a more convenient form by using the result

$$|r_+|^2 - |r_-|^2 = (|\rho_+|^2 - |\rho_-|^2)(|i_+|^2 - |i_-|^2) + (\rho_+^*\rho_- - \rho_+\rho_-^*)(i_+^* i_- - i_+ i_-^*) \tag{3.14}$$

(which follows from equations (3.2)) and the fact that, since T is unitary,

$$\rho_+^*\rho_- - \rho_+\rho_-^* = -2\rho_+\rho_-^*. \tag{3.15}$$

Substitution of equations (3.14) and (3.15) into (3.13) gives

$$S_r = \frac{c}{n\omega}\cot i(|i_+|^2 - |i_-|^2 + |r_+|^2 - |r_-|^2)$$

$$= \frac{c}{n\omega}\cot i(I_+ - I_- + R_+ - R_-) \tag{3.16}$$

in the notation of § 1. This appears to be consistent with the full result of Schilling (1965), as given in equation (35) of his cited work. It also provides correct angular momentum balance: for the case of total internal reflection, equation (1.5) reduces to

$$S' = \frac{F_i}{\omega}\cos^2 i(I_+ - I_- + R_+ - R_-) \tag{3.17}$$

while equation (2.2) reduces to

$$L' = n\frac{F_i}{c}\cos i \sin i\, S, \tag{3.18}$$

and substitution of equation (3.16) in (3.18) provides the required result.

The results of this section thus appear again to support the phase shift calculations (i.e. for the case of total internal reflection with an arbitrarily polarised beam) and the general measure of agreement between various workers seems to be fairly good. However, there is one comparison with the method of Julia and Neveu which is of interest. They suggest that the output beam should consist of two positionally shifted components, namely the eigenstates of the shift matrix $-\mathrm{j}(c/n\omega)T^\dagger\dot{T}$. However, as we point out above in discussing equation (3.8), for the particular case of a circularly polarised input the output beam appears to consist of positionally shifted circularly polarised components, and these are certainly in general not the eigenstates of $-\mathrm{j}(c/n\omega)T^\dagger\dot{T}$. In fact, the usual diagonalisation procedure (together with the fact that, for total internal reflection, ρ_+ and ρ_- differ in phase by $\pi/2$) gives for these eigenstates

$$\begin{bmatrix} |\rho_+| - 1 \\ -\mathrm{j}|\rho_-| \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\mathrm{j}|\rho_-| \\ |\rho_+| - 1 \end{bmatrix} \tag{3.19}$$

which transform under T to

$$\begin{bmatrix} 1 - |\rho_+| \\ -\mathrm{j}|\rho_-| \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\mathrm{j}|\rho_-| \\ 1 - |\rho_+| \end{bmatrix} \tag{3.20}$$

respectively; these then represent the polarisations of the output beam shift eigenstates.

The axes of the polarisation ellipses of all the states represented in expressions (3.19) and (3.20) are inclined at 45° to the principal axes of the reflecting interface. Other features are best studied by reference to the Poincaré sphere (see, e.g., Shurcliff 1962). The input states are situated at such a distance from the poles of the sphere (representing the circularly polarised states) that the retardation effect of the reflection moves each across the corresponding pole to an output state having the same polar distance as the input state. Such states possess to first order the desired property that a rotation followed by the (constant) retardation and a second, equal, rotation transform to a final state which is independent of the size of the rotation. Now, the superposition of two such orthogonal states with fixed amplitude but variable relative phase results in a state lying on a small circle on the Poincaré sphere, with the plane of the circle normal to the diameter joining the orthogonal states, and the position on the circle of the resultant state depending on the relative phase between the component states. In the case of incident circularly polarised light, the output state (equation (3.8)) actually traverses a circle of constant latitude on the sphere (as θ and hence φ_+ vary), while the corresponding superposition of shift eigenstates traverses a circle tangent to the former circle but passing through one of the poles. The two cases coincide exactly only for $\rho_- = 0$, which occurs at critical incidence, when the shift eigenstates are states of circular polarisation. This sort of correspondence is satisfactory in discussing average shifts, but suggests that the detailed structure of the output state cannot in general be handled satisfactorily by either theory, in agreement with the remarks of Ashby and Miller (1976).

4. Partial reflection

The case of partial reflection was treated by Schilling (1965) but has been largely neglected since. It is clear that the simple formulation adopted above extends immediately to the reflected beam and, with a little care, to the transmitted beam. For the latter, the transformation matrix takes the form

$$T_t = \begin{bmatrix} \exp(j\varphi_t) & 0 \\ 0 & \exp(-j\varphi_t) \end{bmatrix} \begin{bmatrix} \tau_+ & \tau_- \\ \tau_- & \tau_+ \end{bmatrix} \begin{bmatrix} \exp(j\varphi_i) & 0 \\ 0 & \exp(-j\varphi_i) \end{bmatrix} \tag{4.1}$$

where consideration of the geometry (using an argument similar to that leading to equation (3.5)) gives

$$\varphi_i = \theta' \frac{\cos i}{\sin i'} \tag{4.2}$$

$$\varphi_t = -\theta' \cot i \tag{4.3}$$

in which θ' represents the angular displacement of wavevector in the transmission medium. Multiplying out the matrices in equation (4.1) gives

$$T_t = \begin{bmatrix} \tau_+ \exp(j\varphi_+) & \tau_- \exp(-j\varphi_-) \\ \tau_- \exp(j\varphi_-) & \tau_+ \exp(-j\varphi_+) \end{bmatrix} \tag{4.4}$$

in which

$$\varphi_+ = \varphi_i + \varphi_t = \theta'(\cos i - \cos i')/\sin i' \tag{4.5}$$

$$\varphi_- = \varphi_i - \varphi_t = \theta'(\cos i + \cos i')/\sin i'. \tag{4.6}$$

The transverse positional displacement can be related to rate of change of phase shift by

$$S_t = \frac{c}{n'\omega} \dot{\varphi} \tag{4.7}$$

where the dot now represents differentiation with respect to θ'.

In applying these results, however, one difficulty presents itself; for neither reflection nor transmission is T unitary, and the corresponding shift matrix $-\mathrm{j}(c/n\omega)T^{\dagger}\dot{T}$ is no longer Hermitian. This suggests that the predicted values of S_r and S_t may be complex; such a non-physical result corresponds to the fact that differently directed plane wave components of the beam may differ in the relative intensities transmitted and reflected. If we ignore this difficulty for the present, formal results for the average shifts (following the same argument as for equation (3.12)) may be written as

$$S_r = -\mathrm{j}\frac{c}{n\omega}\frac{[i_+^* \, i_-^*]T_r^{\dagger}\dot{T}_r\begin{bmatrix} i_+ \\ i_- \end{bmatrix}}{|r_+|^2+|r_-|^2} \tag{4.8}$$

$$S_t = -\mathrm{j}\frac{c}{n'\omega}\frac{[i_+^* \, i_-^*]T_t^{\dagger}\dot{T}_t\begin{bmatrix} i_+ \\ i_- \end{bmatrix}}{|t_+|^2+|t_-|^2} \tag{4.9}$$

where the dot denotes differentiation with respect to θ for S_r, to θ' for S_t, and where

$$\begin{bmatrix} r_+ \\ r_- \end{bmatrix} = T_r\begin{bmatrix} i_+ \\ i_- \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} t_+ \\ t_- \end{bmatrix} = T_t\begin{bmatrix} i_+ \\ i_- \end{bmatrix}. \tag{4.10}$$

Following a procedure similar to that for equation (3.13), these may be evaluated to give

$$S_r = 2\frac{c}{n\omega}\cot i[|\rho_+|^2(|i_+|^2-|i_-|^2) - \rho_+\rho_-^*(i_+^* i_- - i_+ i_-^*)]\frac{1}{|r_+|^2+|r_-|^2} \tag{4.11}$$

and

$$\begin{aligned} S_t = \frac{c}{n\omega}\frac{1}{\sin i}\{&\cos i[(|\tau_+|^2+|\tau_-|^2)(|i_+|^2-|i_-|^2) - (\tau_+\tau_-^* + \tau_+^*\tau_-)(i_+^* i_- - i_+ i_-^*)] \\ &-\cos i'[(|\tau_+|^2-|\tau_-|^2)(|i_+|^2-|i_-|^2) \\ &-(\tau_+\tau_-^* - \tau_+^*\tau_-)(i_+^* i_- - i_+ i_-^*)]\}\frac{1}{|t_+|^2+|t_-|^2} \end{aligned} \tag{4.12}$$

so that S_r and S_t indeed take complex values, unless either $i_+ = 0$ or $i_- = 0$ (in which case the intensity variation mentioned above cannot occur). In this special case, the result for S_r is again consistent with that of Schilling. A formal result for L' may be obtained by substituting (4.11) and (4.12) into equation (2.2), and using $R_+ = |r_+|^2$, etc. This gives

$$\begin{aligned} L' = \frac{F_i}{\omega}\{&(I_+ - I_-)[2|\rho_+|^2\cos^2 i + (|\tau_+|^2+|\tau_-|^2)\cos i\cos i' - (|\tau_+|^2-|\tau_-|^2)\cos^2 i'] \\ &-(i_+^* i_- - i_+ i_-^*)[2\rho_+\rho_-^*\cos^2 i + (\tau_+\tau_-^* + \tau_+^*\tau_-)\cos i\cos i' \\ &-(\tau_+\tau_-^* - \tau_+^*\tau_-)\cos^2 i']\}. \end{aligned} \tag{4.13}$$

A reduction is possible by introducing the requirement for energy balance in the reflection process:

$$R_+ + R_- + \frac{\cos i'}{\cos i}(T_+ + T_-) = 1 \tag{4.14}$$

which, on being written out in terms of the reflection and transmission parameters, imposes on these parameters the conditions

$$|\rho_+|^2 + |\rho_-|^2 + \frac{\cos i'}{\cos i}(|\tau_+|^2 + |\tau_-|^2) = 1 \tag{4.15}$$

and

$$\rho_+\rho_-^* + \rho_+^*\rho_- + \frac{\cos i'}{\cos i}(\tau_+\tau_-^* + \tau_+^*\tau_-) = 0. \tag{4.16}$$

Substitution of (4.15) and (4.16) into (4.13), and use of (3.14) with the equivalent result for transmission, then shows that L' in fact has a real value in all cases, and can be written in the form

$$L' = \frac{F_i}{\omega}[(I_+ - I_-)\cos^2 i + (R_+ - R_-)\cos^2 i - (T_+ - T_-)\cos^2 i']. \tag{4.17}$$

Hence only for the special case of circularly polarised incident light does the simple theory provide a firm prediction for the transverse positional shifts; but even in cases for which the treatment yields non-physical values for the positional shifts, the formal value of L' is in accordance with angular momentum balance. It is, of course, clear that the theory for partial reflection is much less satisfactory than for total reflection, but it does appear that the cases we are unable to treat adequately are those for which the output beam structures are simply not adequately described by a straightforward transverse shift.

One other approach is possible here: one may calculate phase shifts and associated transverse displacements for the input and output spaces separately for each beam and obtain the corresponding total shift by addition. This can be done using equations (3.5) and (4.3) for the phase shifts involved and leads to a very simple final form

$$S_r = \frac{c}{n\omega}\cot i\left(I_+ - I_- + \frac{R_+ - R_-}{R_+ + R_-}\right) \tag{4.18}$$

$$S_t = \frac{c}{n\omega}\frac{1}{\sin i}\left((I_+ - I_-)\cos i - \frac{T_+ - T_-}{T_+ + T_-}\cos i'\right). \tag{4.19}$$

Moreover, use of the general energy conservation result (4.14) ensures angular momentum conservation in all cases. This value of S_r is consistent with that of Schilling. Despite these attractions, it seems to be difficult to justify the line of reasoning used, or indeed to escape the conclusion that a simple shift is not a valid description in the general case. The question of angular momentum balance in this case seems to require a more sophisticated approach.

Finally, the application of an energy flux approach to the case of partial reflection will be briefly discussed. At first sight it is not obvious that such an approach leads to any transverse shift for partial reflection. However, as shown by Imbert (1972), there is a transverse component of Poynting's vector in the region of overlap of incident and reflected beams. Although this component is oscillatory in the direction normal to the surface, its integral over the overlap region (roughly triangular in cross section) will be quite definite for beams more than a few wavelengths in width. This integral is zero for the case of total internal reflection, but turns out not to be so for partial reflection. Moreover, it shows the expected dependence on polarisation. However, one cannot extract separate values for S_r and S_t; in fact one can argue that the combination obtained is just that appearing in L'. Once again, the value of L' in no (non-trivial) case agrees with the phase shift value or the requirement of angular momentum balance.

5. Conclusions

Despite the considerable volume of theoretical work on transverse shifts, and the considerations of this paper, the situation remains confused and the experimental background remains meagre. It would seem highly desirable to measure true transverse shifts in situations of varying angle of incidence and polarisation, including the case of partial reflection.

Acknowledgment

I would like to thank Professor R V Jones for bringing this problem to my attention.

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